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LETTER TO THE EDITOR

Ternary vector cross products

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Abstract. Ternary vector cross products are studied in their own right. Results include a new proof of Hurwitz's theorem and a 'principle of duplicity'. Upon breaking the symmetry in eight dimensions, by choosing a preferred axis, this last principle implies the well known triality principle for octonions and SO(8) transformations. In displaying canonical forms it helps to put the eight basis vectors in correspondence with the eight points of the three-dimensional affine geometry over F_2.

Let E be a real n-dimensional vector space equipped with a positive definite inner product $\langle\ ,\ \rangle$. For $n \geqslant 3$ a *ternary vector cross product* for E is defined to be a map $X : E^3 \to E$ which satisfies the axioms (cf [1, 2]):

(i) X is trilinear

(ii) $X(a, b, c)$ is orthogonal to each of a, b, c (1)

(iii) $\|X(a, b, c)\| = \|a \wedge b \wedge c\|$.

Here $\|a\|^2 = \langle a, a\rangle$ and $\|a_1 \wedge \ldots \wedge a_r\|^2$ denotes the Gram determinant whose i, j entry is $\langle a_i, a_j\rangle$. Geometrically speaking axiom (iii) asserts that the length of $X(a, b, c)$ is equal to the volume of the parallelepiped determined by the vectors a, b, c. A real inner product space E, dim $E = n \geqslant 3$, which has been equipped with a preferred ternary vector cross product, will be termed a 3Xn algebra. By modifying the axioms in the obvious way we obtain the definition of a binary vector cross product $X(a, b)$, and so of a 2Xn algebra, $n \geqslant 2$. More generally, for $2 \leqslant r \leqslant n$ we have the notion of a rXn algebra.

Each 3Xn algebra E can be converted into a *ternary composition algebra*, referred to as the associated 3Cn algebra, by defining a ternary multiplication $\{\ \} : E^3 \to E$ by

$$\{abc\} = \langle a, b\rangle c + \langle b, c\rangle a - \langle c, a\rangle b + X(a, b, c) \qquad (2)$$

and noting that it satisfies the property

$$\|\{abc\}\| = \|a\|\,\|b\|\,\|c\|. \qquad (3)$$

(The near-miraculous cancellation of terms involved in this check can be given an explanation in terms of the Clifford algebra of E.) Observe that for each choice of unit vector $e \in E$ we can obtain a 2Cn algebra, with e as identity, out of our 3Xn algebra by defining

$$ac = \{aec\}. \qquad (4)$$

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Similarly, setting E' to be the subspace of E orthogonal to e, we can obtain a $2\mathrm{X}(n-1)$ algebra out of our $3\mathrm{X}n$ algebra by defining $X(x, z) = X(x, e, z)$ for $x, z \in E'$. Also associated with a $3\mathrm{X}n$ algebra is the scalar quadruple product Φ defined by

$$\Phi(a, b, c, d) = \langle a, X(b, c, d)\rangle. \tag{5}$$

Axioms (i) and (ii) entail that Φ is alternating, whence so is X.

We denote by $X \to {}^A X$ the natural action of $A \in \mathrm{O}(E)$ upon a ternary vector cross product X. Thus ${}^A X(a, b, c) = AX(A^{-1}a, A^{-1}b, A^{-1}c)$. Equivalently, in terms of Φ viewed as an element of $\bigwedge^4 E$, we have ${}^A\Phi = (\wedge^4 A)\Phi$. The automorphism group Aut E of a $3\mathrm{X}n$ algebra E consists of the isotropy group of X: Aut $E = \{A \in \mathrm{O}(E)\colon {}^A X = X\}$.

For $n = 4$ we can make E into a 3X4 algebra in precisely two ways, arrived at by taking Φ in (5) to be Δ or $-\Delta$, where Δ is a normalised determinant function for E. Similarly in dimension n there are precisely two $(n-1)\mathrm{X}n$ algebras. Since ${}^A\Delta = (\det A)\Delta$ we see in these cases that Aut $E = \mathrm{SO}(E) \simeq \mathrm{SO}(n)$. Of course these $(n-1)\mathrm{X}n$ algebras are the obvious generalisations to n dimensions of the familiar 2X3 algebras (one 'right-handed', the other 'left-handed'). Indeed (see [3]) almost all the properties of a 2X3 algebra receive straightforward generalisations to a $(n-1)\mathrm{X}n$ algebra. The question arises, do there exist $r\mathrm{X}n$ algebras, $2 \leq r \leq n$, other than the $(n-1)\mathrm{X}n$ algebras? The answer is known (see [1, 2]) although perhaps a little surprising: there do exist 'exceptional' $r\mathrm{X}n$ algebras, but only of the kinds 2X7 and 3X8. The existence of these two kinds of exceptional algebras can be related to the existence of the (not associative) 2C8 algebra $\mathbb{O}$ of the octonions—see, for example, [2, 4, 5]—and usually their properties are obtained by appeal to those of the octonions.

The purpose of this letter is to study 3X8 algebras in their own right, without appeal to the octonions. There is some modest virtue in so doing, since a 3X8 algebra E is more symmetrical than $\mathbb{O}$, the respective automorphism groups being Spin(7) and G_2 of dimensions 21 and 14. In the present letter it is an easy matter to obtain the less symmetrical from the more symmetrical by making a choice of a(ny) unit vector $e \in E$ to act as the identity element of $\mathbb{O}$, as in (4). In the same vein, rather than appeal as in [2] to Hurwitz's theorem, we will instead obtain a neat proof of this theorem (in the cases $n > 2$) arising directly from considering $3\mathrm{X}n$ algebras in their own right. (Unlike the usual proofs (see, e.g., [6]) our version of Hurwitz's theorem does not require us to make a preferred choice of unit vector $e \in E$.) Incidentally we should point out that this letter confines itself to a purely algebraic investigation; no mention is made of applications to manifolds (cf [4, 7]), nor to any bearing upon recent attempts (cf [8]) at constructing a unified theory of all physical interactions.

Properties of 3Xn algebras. For a $3\mathrm{X}n$ algebra E we define the multiplication operators $T_{a,b}$ by $T_{a,b}c = X(a, b, c)$. They are seen to be skew-adjoint maps $E \to E$, or equally well bivectors—since we make the usual identifications $\mathrm{sk}(E, E) \simeq \bigwedge^2 E \simeq \mathrm{so}(E)$. In fact we make more use of the left and right multiplication operators $\gamma_{a,b}$ and $\sigma_{a,b}$ of the associated $3\mathrm{C}n$ algebra:

$$\gamma_{a,b}c = \{abc\} \qquad \sigma_{a,b}c = \{cba\}. \tag{6}$$

Note that $\gamma_{a,b}$ and $\sigma_{a,b}$ have skew-adjoint parts $2a \wedge b + T_{a,b}$ and $2a \wedge b - T_{a,b}$, respectively, and that both have $\langle a, b\rangle I$ as their self-adjoint part. Some of their properties are set out in the following two lemmas.

Lemma A.
(a) $\gamma_{a,b}\gamma_{b,a} = \|a\|^2\|b\|^2 I$, for all $a, b \in E$
(b) if $\|a\| = \|b\| = 1$, then $\gamma_{a,b} \in \mathrm{SO}(E)$
(c) if $\langle a, b\rangle = 0$, then $\gamma_{a,b} = -\gamma_{b,a} \in \mathrm{sk}(E, E)$.

Lemma B. Let $\{b, c, d\}$ denote any choice of ordered orthonormal triad of vectors of the 3Xn algebra E, and set $a = \{b, c, d\}$ $(= X(b, c, d))$. Let $H = <a, b, c, d>$ (the subspace spanned by a, b, c, d) and define the involution $\Pi^H \in \mathrm{O}(E)$ to be +1 on H and -1 on $H^\perp$. Then
(a) $\gamma_{b,c}$ anticommutes with $\gamma_{b,d}$,
(b) H is a 3X4 subalgebra of E having $\{a, b, c, d\}$ as a positive orthonormal basis (i.e. $\Phi(a, b, c, d) = +1$),
(c) for non-zero $h, k \in H$ and non-zero $p \in H^\perp$, $\gamma_{h,k}$ maps H onto H and $H^\perp$ onto $H^\perp$ (and so commutes with Π^H), whilst $\gamma_{p,h}$ injects H into $H^\perp$,
(d) $\gamma_{a,b}\gamma_{a,c}\gamma_{a,d} = \Pi^H = -\sigma_{a,b}\sigma_{a,c}\sigma_{a,d}$.

Proofs. The chief weapon is to use (3) in its various polarised forms. For example, linearising (3) in the vector c yields (a) of lemma A. A subsidiary weapon is the fact that we know all concerning a 3X4 algebra. In particular, for a 3X4 algebra with positive orthonormal basis $\{a, b, c, d\}$, we can check that

$$\gamma_{a,b}\gamma_{a,c}\gamma_{a,d} = I = -\sigma_{a,b}\sigma_{a,c}\sigma_{a,d}$$

and use this in the proof of (d) of lemma B.

Remark. Lemma A, and (a) and (c) of lemma B, hold if $\gamma_{a,b}$ is replaced by $\sigma_{a,b}$.

Theorem C. For a 3Xn algebra either $n = 4$ or $n = 8$.

Proof. By axioms (ii) and (iii) we have $n \geqslant 4$. For $n > 4$ choose a 3X4 subalgebra $H = <a, b, c, d>$ as in lemma B and let p be any non-zero element of $H^\perp$. Then $\gamma_{a,p}$ is an invertible operator which, by lemma B, anticommutes with Π^H. Hence Π^H has zero trace, whence $\dim H^\perp = \dim H = 4$, and so $n = 8$.

Remark. Given a 2Cn algebra E with $n > 2$ we can suppose it has an identity element e and corresponding conjugation $K : a \mapsto \bar{a}$, where

$$Ka = \bar{a} = 2\langle a, e\rangle e - a. \tag{7}$$

Upon defining $X(a, b, c)$ by (2) with $\{abc\}$ taken to be $(a\bar{b})c$, we can check that E is thereby converted into a 3Xn algebra. Consequently Hurwitz's theorem for 2Cn algebras follows, in the cases $n > 2$, from theorem C.

Remark. We will make use of the map $K \in \mathrm{O}_-(E)$ later on for a 3X8 algebra E. In ternary notation note that $K = \mu_{e,e}$ where $\mu_{a,c}$ is the 'middle multiplication' operator $b \mapsto \{abc\}$.

3X8 algebras. A pleasing feature of a 3X8 algebra E is that each 3X4 subalgebra comes along with a 'partner' $P = H^\perp$ which is also a 3X4 subalgebra.

Theorem D. If H is a 3X4 subalgebra of a 3X8 algebra E, then so is $P = H^\perp$.

Proof. Consider the identity $\langle h, \{pqr\}\rangle = \langle \sigma_{h,p}q, r\rangle$. Setting $h \in H$ and $p, q, r \in P$ our previous results entail that the RHS is zero. Hence $\{pqr\} \in P$.

Theorem E. Ternary vector cross products on E, $\dim E = 8$, fall into two O(8) orbits: type I $\cup$ type II. If $X \in$ type I then $-X \in$ type II. Each O(8) orbit splits into two SO(8) orbits, say $\mathrm{I}^{\mathrm{R}} \cup \mathrm{I}^{\mathrm{L}}$ and $\mathrm{II}^{\mathrm{R}} \cup \mathrm{II}^{\mathrm{L}}$.

Proof. For $E = H \oplus P$ as in theorem D choose unit vectors $h \in H$, $p \in P$ and note that $\sigma_{p,h}$, $\gamma_{p,h}$ are isometries which map H onto P (and P onto H). A dichotomy arises: either $\gamma_{p,h}$ (type I, say) or $\sigma_{p,h}$ (type II, say) maps a positive orthonormal basis for H onto a positive orthonormal basis for P. Replacing X by $-X$ amounts to the interchange of $\sigma_{p,h}$ and $\gamma_{p,h}$. Finally X defines an orientation of E via a positive basis for H together with a positive basis for P.

From now on we consider 3X8 algebras of type I. For such an algebra E the following is true: choose any 3X4 subalgebra H and choose any unit vectors $h \in H$, $p \in P = H^{\perp}$; then $\gamma_{p,h}$, suitably restricted, defines an isomorphism of the 3X4 algebras H and P. (This follows from the foregoing dichotomy upon using a continuity argument.) A canonical form for the 3X8 algebra E can now be obtained, with respect to a *canonical orthonormal basis* $\{e_a\,;\ a = 0, 1, 2, 3, 0', 1', 2', 3'\}$ constructed as follows. Choose any othonormal triad $\{e_1, e_2, e_3\}$, set $e_0 = \{e_1 e_2 e_3\}$ and, for any choice of unit vector $e_{0'} \in \langle e_0, e_1, e_2, e_3 \rangle^{\perp}$, define $e_{i'} = \{e_{0'} e_0 e_i\}$. The fact that all eight canonical axes enter democratically into the ensuing canonical form can be highlighted by putting the eight basis vectors e_a in a one-to-one correspondence with the eight points denoted, say, by

$$0,\ 1,\ 2,\ 3,\ 0',\ 1',\ 2',\ 3'$$

of the finite three-dimensional affine geometry $\mathscr{A}$ over the field F_2 of order 2. In this geometry there are seven quadruples of mutually parallel lines (each line consisting of two points) which we take to be

$$\begin{array}{llll} 01 & 23 & 3'2' & 1'0' \\ 02 & 31 & 1'3' & 2'0' \\ 03 & 12 & 2'1' & 3'0' \\ 00' & 11' & 22' & 33' \\ 01' & 3'2 & 32' & 0'1 \\ 02' & 1'3 & 13' & 0'2 \\ 03' & 2'1 & 21' & 0'3 \end{array} \tag{8}$$

and seven pairs (λ, λ^*) of parallel planes

$$\begin{array}{lllllll} \lambda\ = 0123 & 011'0' & 022'0' & 033'0' & 013'2' & 021'3' & 032'1' \\ \lambda^* = 0'1'2'3' & 233'2' & 311'3' & 122'1' & 0'1'32 & 0'2'13 & 0'3'21. \end{array} \tag{9}$$

(One can view the eight points of $\mathscr{A}$ as the vertices of a cube in three dimensions, with for example 0123 and 0'1'2'3' as the inscribed 'tetrahedra'. However, one has to remember that these 'tetrahedra' are in fact planes. Moreover one has to remember that the two 'diagonals' of each square face are parallel.)

For the purposes of displaying our canonical forms a further refinement is necessary: the two orderings *ab* and *ba* of the points of a line will be distinguished, and we will view the lines of a quadruple in (8) as 'strictly parallel' (rather than 'antiparallel'), which we write as '$\sim$', when the order is as displayed. So, for example, we have

$$00' \sim 11' \sim 22' \sim 33'. \tag{10}$$

The scheme (8) is consistent with '$\sim$' being an equivalence relation which obeys the following rule:

$$\text{if } a, b, c, d \text{ are distinct, then } ab \sim cd \text{ implies } ad \sim bc. \tag{11}$$

In fact, using this rule, the whole scheme (8) follows from, for example, (10) combined with, for example, $01 \sim 23$. Moreover the order in which we write the points $abcd$ of each plane in (9) is determined, up to an even permutation, by the requirement $ab \sim cd$.

In terms of the multiplication operators $\gamma_{u,v}$ our canonical form is seen to be given by

$$\gamma_{00'} = \gamma_{11'} = \gamma_{22'} = \gamma_{33'} = -(J_{00'} + J_{11'} + J_{22'} + J_{33'}) \tag{12}$$

together with six other analogous equations read off from the scheme (8). Here we have set $J_{ab} = -2e_a \wedge e_b$ and have also, for $u = e_a$, $v = e_b$, written $\gamma_{u,v}$ as γ_{ab}. (Incidentally, in the notation of lemma B, we have the properties $\gamma_{a,b} = \gamma_{c,d}$ and $\sigma_{a,b}\sigma_{c,d} = \Pi^H$. Moreover $\sigma_{u,v}\sigma_{v,w} = \langle v, v\rangle \sigma_{u,w}$ holds for all $u, v, w \in E$.)

In terms of $\Phi_{abcd} \equiv \Phi(e_a, e_b, e_c, e_d)$ our canonical form asserts that Φ_{abcd} equals $+1(-1)$ whenever $abcd$ is an even (odd) permutation of one of the fourteen values in (9), and equals zero otherwise. Considering Φ as an element of $\bigwedge^4 E$ we have $\Phi = \Sigma_\lambda (\Phi^\lambda + \Phi^{\lambda^*})$ where, for example, if $\lambda = 0123$ then $\Phi^\lambda = e_0 \wedge e_1 \wedge e_2 \wedge e_3$ and $\Phi^{\lambda^*} = e_{0'} \wedge e_{1'} \wedge e_{2'} \wedge e_{3'}$. Granted that we define the star operator $\bigwedge^4 E \to \bigwedge^4 E$ with respect to the orientation on E defined (see the proof of theorem E) by X, observe that Φ is self-dual: $*\Phi = +\Phi$. Observe also that for any choice of unit vector $e \in E$ we can express $\Phi \in \bigwedge^4 E$ in the form

$$\Phi = e \wedge \phi + *_7 \phi \tag{13}$$

where $\phi \in \bigwedge^3 E'$, $E' = \langle e \rangle^\perp$ and $*_7$ is a star operator $\bigwedge^3 E' \to \bigwedge^4 E' \subset \bigwedge^4 E$.

Each pair (λ, λ^*) of parallel affine planes in (9) is associated with a decomposition $E = H^\lambda \oplus (H^\lambda)^\perp$ of the 3X8 algebra E into two 3X4 subalgebras. Thus for $\lambda = 0123$ we have $H^\lambda = \langle e_0, e_1, e_2, e_3 \rangle$ and $H^{\lambda^*} = (H^\lambda)^\perp = \langle e_{0'}, e_{1'}, e_{2'}, e_{3'} \rangle$. Equally well the pair (λ, λ^*) is associated with a pair $(\Pi^\lambda, -\Pi^\lambda)$ of involutions $\in \mathrm{SO}(E)$, where $\Pi^\lambda \equiv \Pi^{H^\lambda}$. The seven involutions Π^λ mutually commute, which goes along (see, e.g., [9]) with the fact the fourteen 3X4 subalgebras H^λ, $(H^\lambda)^\perp$ are mutually compatible subspaces of E. Notice that for $\lambda \neq \mu$ we have $\dim(H^\lambda \cap H^\mu) = 2$—corresponding to the fact that the distinct non-parallel affine planes λ, μ intersect in the two points of an affine line.

We move now to some results concerning SO(8) and triality. Let M denote the self-adjoint operator on $\bigwedge^2 E \simeq \mathrm{sk}(E, E) \simeq \mathrm{so}(E) \simeq \mathrm{so}(8)$ which is defined by $8a \wedge b \mapsto \sigma_{a,b} - \sigma_{b,a}$. Then one shows that $M^2 = I$ and that M is an outer automorphism of the Lie algebra so(8). Define $N = {}^K M$, i.e. $N = LML$ where $L = \wedge^2 K : B \to KBK$. (If M arises from $X \in$ type I^{R}, N arises in the same manner from ${}^K X \in$ type I^{L}.) One finds that $LM = MN = NL$ $(=\Omega$, say), whence $\Omega^2 = ML = NM = LN$ and $\Omega^3 = I$. Then $\{I, L, M, N, \Omega, \Omega^2\} \simeq S_3$ forms a group of outer automorphisms of so(8).

Theorem F ('principle of duplicity'). Given any $A \in \mathrm{SO}(8)$ there are precisely two SO(8) transformations $\pm A'$ such that $A\{abc\} = \{AaA'bA'c\}$.

Proof. For any automorphism Θ of the Lie algebra $\mathrm{sk}(E, E) \simeq \mathrm{so}(8)$ the local automorphism θ of the Lie group $\mathrm{SO}(E) \simeq \mathrm{SO}(8)$, defined locally by $\theta(\exp B) = \exp \Theta B$, satisfies

$$\Theta \circ \mathrm{Ad}\, A = \mathrm{Ad}(\theta(A)) \circ \Theta \qquad \text{for } A \text{ in some neighbourhood of } I \text{ in } \mathrm{SO}(8). \tag{14}$$

Setting $\Theta = M$ we evaluate (14) upon the bivector $c \wedge b$, then evaluate the resulting skew-adjoint maps upon $a \in E$ to obtain

$$A\{abc\} = \{Aa\, m(A)b\, m(A)c\} \qquad \text{(locally)}. \tag{15}$$

Remark. If we pass to the octonionic binary multiplication as in (4) (thereby losing symmetry) the above 'principle of duplicity' is seen to imply the well known (see [10, 11]) triality principle. For (15) yields the result $A(ac) = (n(A)a)(m(A)c)$, valid locally, or equivalently, noting that $l(A) = KAK$

$$\overline{A(\overline{ac})} = (\omega(A)a)(\omega^2(A)c) \qquad \text{(locally)}.$$

Remark. A possible definition of Spin(8) is as that subgroup of SO(8) × SO(8) which consists of all duplicity pairs (A, A'), i.e. as in theorem F. The covering homomorphism Spin(8)$\mapsto$SO(8) is $(A, A') \mapsto A$ and has kernel $\cong Z_2$ consisting of (I, I), $(I - I)$, which last are seen to be connected in Spin(8) as defined.

Remark. As is well known, the automorphisms L, M, N keep fixed subalgebras so(7), spin$^{\mathrm{R}}$(7) and spin$^{\mathrm{L}}$(7) of so(8). These three subalgebras of dimension 21 are cyclically permuted by the triality automorphisms Ω, Ω^2, and their common intersection is a subalgebra g_2 of dimension 14. The automorphisms of the 3X8 algebra E, which form a subgroup Spin$^{\mathrm{R}}$(7) of SO(8), are arrived at as the special case $m(A) = \pm A$ of (15).

A typical Spin$^{\mathrm{R}}$(7) transformation is

$$A = R_{00'}(\theta_0)R_{11'}(\theta_1)R_{22'}(\theta_2)R_{33'}(\theta_3) \qquad (\text{with } \theta_0 + \theta_1 + \theta_2 + \theta_3 = 0) \tag{16}$$

where $R_{aa'}(\theta_a)$ denotes the rotation $\exp(\theta_a J_{aa'})$ through an angle θ_a in the oriented plane $< e_a, e_{a'} >$. The condition $\Sigma\, \theta_a = 0$ on the angles θ_a ensures that the generator $\Sigma_{a=0}^{3} \theta_a J_{aa'}$ is orthogonal to the -1 eigenspace of M (which is spanned by the γ_{ab} typified by (12)) and hence lies in the $+1$ eigenspace spin$^{\mathrm{R}}$(7) of M.

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References

[1] Eckmann B 1942-3 *Comment. Math. Helv.* **15** 318
[2] Brown R B and Gray A 1967 *Comment. Math. Helv.* **42** 222
[3] Shaw R 1987 *Int. J. Math. Educ. Sci. Technol.* to be published
[4] Harvey R and Lawson H B 1982 *Acta Math.* **148** 47
[5] Dündarer R, Gürsey F and Tze C-H 1984 *J. Math. Phys.* **25** 1496
[6] Jacobson N 1974 *Basic Algebra I* (San Francisco: Freeman)
[7] Gray A 1969 *Trans. Am. Math. Soc.* **141** 465
[8] Duff M J, Nilsson B E W and Pope C N 1986 *Phys. Rep.* **130** 1
[9] Shaw R 1982 *Linear Algebra and Group Representations* vol 1 (New York: Academic)
[10] Van Der Blij F and Springer T A 1960 *Niew. Arch. Wiskunde* **VIII** 158
[11] Günaydin M and Gürsey F 1973 *J. Math. Phys.* **14** 1651